On Minkowski dimension of quasicircles

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Abstract

1 Introduction.

Let $\Omega \subsetneq \mathbb{C}$ be a simply connected domain containing 0: by the Riemann Mapping theorem, there is a unique conformal map f from the unit disk $\mathbb{D} = \{|z| < 1\}$ onto Ω such that $f(0) = 0, f'(0) > 0$. In this paper we are interested in domains with fractal boundary and more precisely with the Hausdorff dimension of these boundaries. Well-known examples of fractal curves which have deserved a lot of investigations and attentions are the Julia sets and the limit sets of quasifuchsian groups because of their dynamical properties.
For instance, let us consider the family of quadratic polynomials

$$P_t(z) = z^2 + t,\ t \in \mathbb{C}$$

in the neighborhood of $t = 0$. There is a smooth family of conformal map ϕ_t from $\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$ onto the basin of infinity of the polynomial $P_t(z)$ (the component containing ∞ of its Fatou set) with $\phi_0(z) = z$ and conjugating P_0 to P_t on their basins of infinity. We thus have:

$$\phi_t(P_0) = P_t(\phi_t(z)),\ z \in \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}. \tag{1}$$

Each ϕ_t extends to a quasiconformal map on the sphere $\overline{\mathbb{C}}$. Taking the derivative of the equation (1) with respect to t, we obtain the equation:

$$\dot{\phi}_t(z^2) = 2\phi_t(z)\dot{\phi}_t(z) + 1, \tag{2}$$

where $\dot{\phi}_t = \dfrac{\partial \phi}{\partial t}$. Let $V(z)$ denote the holomorphic vector field of $V(z) = \left.\dfrac{\partial \phi_t}{\partial t}\right|_{t=0}$. Letting $t = 0$ in the equation (2), we get that the holomorphic vector field V satisfies the functional equation:

$$V(z^2) = 2zV(z) + 1. \tag{3}$$

If we replace z by z^2 in the preceding equation, we obtain that

$$V(z^4) = 2z^2V(z^2) + 1. \tag{4}$$

Injecting $V(z^2)$ in (3) into (4), one gets $V(z) = -\left(\dfrac{1}{2z} + \dfrac{1}{2z2z^2}\right) + \dfrac{V(z^4)}{2z2z^2}$. And by induction we can obtain $V(z) = -\displaystyle\sum_{k=1}^{n-1} \frac{1}{2z2z^2...2z^{2^k}} + \frac{V(z^{2^n})}{2z2z^2...2z^{2^{n-1}}}$. The term $\dfrac{V(z^{2^n})}{2^{n+1}z^{2^n-1}}$ tends to 0 as n tends to ∞. Therefore $V(z)$ can be written as an infinite sum

$$V(z) = -z\sum_{k=0}^{\infty} \frac{1}{2^{k+1}z^{2^{k+1}}}. \tag{5}$$

Using thermodynamic formalism, Ruelle [Rul] (see also [Zin] and [McM]) proved that

$$\frac{d^2}{dt^2}\mathrm{H.dim}(J(P_t))\Big|_{t=0} = \lim_{r\longrightarrow 1}\frac{1}{4\pi}\frac{1}{\log\frac{1}{1-r}}\int_{|z|=r}|v'(z)|^2|dz|. \tag{6}$$

Using then the explicit formula (5) of V, he could proved that

$$\mathrm{H.dim}(J(P_t)) = 1+\frac{|t|^2}{4\log 2}+o(|t|^2). \tag{7}$$

for this particular family.
Passing to the disc instead of its complement, let us consider a general analytic one-parameter family (ϕ_t), $t\in U$, a neighborhood of $t=0$, of conformal maps with $\phi_0 = id$ and $\phi_t(0)=0$, $\forall t\in U$.
Then

$$\phi_t(z) = \int_0^z e^{\log\phi_t'(u)}du$$

and

$$\frac{\partial}{\partial t}\phi_t(z) = \int_0^z \frac{\partial}{\partial t}\left(\log\phi_t'(u)\right)e^{\log\phi_t'(u)}du.$$

From which follows that

$$V(z) = \frac{\partial}{\partial t}\phi_t(z)\Big|_{t=0} = \int_0^z \frac{\partial}{\partial t}\left(\log\phi_t'(u)\right)\Big|_{t=0}du$$

and $b(z) = V'(z) = \frac{\partial}{\partial t}\left(\log\phi_t'(z)\right)\Big|_{t=0}$ belongs to the Bloch space $\mathcal{B}$ which is defined as follows:

$$\mathcal{B} = \left\{\ b \text{ holomorphic in } \mathbb{D};\ \sup_{\mathbb{D}}(1-|z|)|b'(z)|<\infty\ \right\}.$$

It follows from $\lambda-$lemma (see [IT]) that ϕ_t has a quasiconformal extension to the plane if t is small enough. In particular $\Gamma_t = \phi_t(\partial\mathbb{D})$ is well-defined.
In [McM], Mc Mullen asked the following question: Under which condition on the family of (ϕ_t) it is true that

$$\frac{d^2}{dt^2}\mathrm{H.dim}\ (\Gamma_t)\Big|_{t=0} = \lim_{r\longrightarrow 1}\frac{1}{4\pi|\log(1-r)|}\int_{|z|=r}|b(z)|^2|dz| \quad ? \tag{8}$$

In other words, the question addresses the problem of how much formula (6) owes to dynamical properties.
Conversely, starting from a function $b\in\mathcal{B}$, it is known that if we put

$$\phi_t(z) = \int_0^z e^{tb(u)}du, \quad b\in\mathcal{B}, \tag{9}$$

is an analytic family and there exists a neighborhood U of 0 such that if $t\in U$ then ϕ_t is a conformal map with quasiconformal extension and we denote by Γ_t the image of the unit circle by ϕ_t.
The aim of the work is two-fold: we will first describe a large family of function $b\in\mathcal{B}$ for which if ϕ_t is defined by (9), (t being real) then (8) is true with Hausdorff dimension replaced by Minkowski dimension. This class will be defined in term of the square function of the associated of dyadic martingale of $\mathrm{Re}(b)$. Details and proper statement will be given in 2.

The second result of this paper is a counter-example. The starting point is the construction by Kahane and Piranian of a so-called "non Smirnov" rectifiable domain. These authors have constructed a Bloch function b such that if we consider the associated family (ϕ_t) as in (9) then $\phi_t(\partial\mathbb{D})$ is rectifiable for $t<0$. This function is very singular in the sense that

$$b(z)=\int_0^{2\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}d\mu(\theta),$$

where μ is singular with respect to Lebesgue measure on the circle. We use this feature to prove that there exists $c>0$ such that

$$\text{M.dim}(\Gamma_t)\geq 1+ct^2,\quad t>0\ \text{ small.}$$

which contradicts $\lim\limits_{t\longrightarrow 0}\dfrac{\text{M.dim}(\Gamma_t)-1}{t^2}=0$ by (8) with Hausdorff dimension replaced by Minkowski dimension.

2 Martingale condition

Before giving statement of the first result of this paper, we recall some preliminaries on Bloch function and the notion of dyadic martingale.

2.1 Preliminaries on Bloch function

Proposition 1 *If $b\in\mathcal{B}$ and $b(0)=0$ then*

$$\frac{1}{2\pi}\int_{\mathbb{T}}|b(r\xi)|^{2n}|d\xi|\leq n!\|b\|_{\mathcal{B}}^{2n}\left(\log\frac{1}{1-r^2}\right)^n$$

for $0<r<1$ and $n=0,1,...$

Proof: See [Pom].
This proposition implies that if $b\in\mathcal{B}, b(0)=0$,

$$\limsup_{r\longrightarrow 1}\frac{\int_0^{2\pi}|b(re^{i\theta})|^2d\theta}{2\pi\log(\frac{1}{1-r})}\leq\|b\|_{\mathcal{B}}^2<+\infty. \tag{10}$$

This proposition can be generalized as follows.

Corollary 1 *If $b\in\mathcal{B}$ and $b(0)=0$ then there exists a constant C such that*

$$\int_{\mathbb{T}}|b(r\xi)|^p|d\xi|\leq C\left(\log\frac{1}{1-r^2}\right)^{p/2}$$

for $0<r<1$ and $p>0$.

Proof: For $p>0$, there exists a positive integer n such that $0<\dfrac{p}{2n}<1$. Applying the Hölder's inequality for $\alpha=\dfrac{p}{2n}<1$, we deduce that

$$\int_{\mathbb{T}}|b(r\xi)|^{2n}|d\xi|\geq\left(\int_{\mathbb{T}}|b(r\xi)|^{2n\alpha}\right)^{1/\alpha}\left(2\pi\right)^{1/\beta},$$

where $\frac{1}{\alpha}+\frac{1}{\beta}=1$. Then Proposition 1 implies that

$$\left(\int_{\mathbb{T}}|b(r\xi)|^p|d\xi|\right)=\left(\int_{\mathbb{T}}|b(r\xi)|^{2n\alpha}|d\xi|\right)\leq(2\pi)^{-\alpha/\beta}\left(\int_{\mathbb{T}}|b(r\xi)|^{2n}|d\xi|\right)^{\alpha}\leq C\left(\log\frac{1}{1-r^2}\right)^{p/2},$$

where $C=(2\pi)^{-\alpha/\beta}(n!\|b\|_{\mathcal{B}}^{2n})^{\alpha}$.
Let h be a (complex-valued) continuous function on the unit circle $\mathbb{T}$ and satisfying

$$\sup_{|z|=1}|h(e^{i\theta}z)-2h(z)+h(e^{-i\theta}z)|\leq C\theta,\quad \text{for } \theta>0.$$

This function is called a Zygmund function.

Theorem 1 ***(Zygmund)*** *Let b be analytic on the disk $\mathbb{D}$ and let $h(z)$ be a primitive function of b. Then b belongs to Bloch space $\mathcal{B}$ if and only if h is continuous on the closed disk $\overline{\mathbb{D}}$ and h is a Zygmund function.*

Proof: See [Dur].
Let $I=(e^{i\theta_1},e^{i\theta_2})$ be a subarc of $\partial\mathbb{D}$. We can define b_I, the mean value of b on the arc $I\subset\partial\mathbb{D}$, as the limit $\lim_{r\longrightarrow 1}(b_r)_I$, where $b_r(z)=b(rz), z\in\mathbb{D}$. Integration by parts shows that

$$b_I=\lim_{r\longrightarrow 1}\frac{1}{|I|}\int_I b(re^{i\theta})d\theta=\frac{-ie^{-i\theta_2}h(e^{i\theta_2})+ie^{-i\theta_1}h(e^{i\theta_1})}{|I|}+\frac{1}{|I|}\int_I e^{-i\theta}h(e^{i\theta})d\theta$$

and by the property of continuity up to the boundary of the primitive function $h(z)$, the limit exists. Hence the definition of mean value of Bloch function is well-defined. We recall now the notion of dyadic martingale of a Bloch function.

2.2 Dyadic martingale.

On the probability space $(\partial\mathbb{D},|.|)$ $(|d\xi|=d\theta/2\pi,\ \xi=e^{i\theta}\in\partial\mathbb{D})$, we consider the increasing sequence of σ-algebras $\{\mathcal{F}_n,n\geq 0\}$ generated by the partitions of the unit circle by the intervals bounded by the (2^n)th roots of the unity.
Let b be a Bloch function, $b(0)=0$. We defined $S=(S_n,\mathcal{F}_n)$ by setting $S_n|I=b_I$ on each dyadic interval I of rank n. In other words $S_n=\mathbf{E}(b|\mathcal{F}_n)$. Then

$$\forall\xi\in\partial\mathbb{D},\ \ S_n(\xi)=\sum_{I\in\mathcal{F}_n}b_I\chi_I(\xi).$$

This sequence is a martingale in the sense that $\mathbf{E}(S_{n+1}|\mathcal{F}_n)=S_n$. And it has the property:

$$\forall n,\forall\xi\in\partial\mathbb{D},\qquad\left|\ S_n(\xi)-b((1-2^{-n})\xi)\ \right|\leq C\|b\|_{\mathcal{B}},\tag{11}$$

where C is an absolute constant (see [Mak]).
We consider the increasing sequence $\langle S\rangle_n^2=\sum_{j=1}^n\mathbf{E}((\Delta S_j)^2|\mathcal{F}_{j-1})$, where $\Delta S_j=S_j-S_{j-1}$. In the dyadic case ΔS_j^2 is $\mathcal{F}_{j-1}$ measurable, so that $\langle S\rangle_n^2=\sum_{j=1}^n(\Delta S_j)^2$.
We call $\langle S\rangle_\infty^2=\sum_{k\geq 1}(\Delta S_k)^2$ the square function.
This first result is based on the computing of the integral means $\int_{|z|=r}e^{t\mathrm{Re}(b(z))}|dz|,\ b\in\mathcal{B}$ in which there is only the real part of a Bloch function b that appears, so that we just need the dyadic martingale which arises from a real part of the Bloch function. Let us state this result.

2.3 Statement of Theorem 2.

Let b be a Bloch function and b_n be the dyadic martingale of $\mathrm{Re}(b)$. Let us assume the following condition for its square function $\langle S\rangle_n^2$:

$$\forall\theta\in[0,2\pi],\qquad \left|\langle S\rangle_n^2(e^{i\theta})-\frac{1}{2\pi}\int_0^{2\pi}\langle S\rangle_n^2(e^{i\theta})d\theta\right|\le n\delta(n),\qquad (*)$$

where $\delta(n)$ is a positive function which depends only on n and which tends to zero as n tends to ∞. Let us also write $d(t)=\mathrm{M.dim}(\Gamma_t)$.

Theorem 2 *If b belongs to $\mathcal{B}$ and satisfies the condition $(*)$ then the Minkowski dimension of Γ_t has the following development at zero:*

$$M.dim(\Gamma_t)=1+\limsup_{r\longrightarrow 1}\frac{\int_0^{2\pi}|b(re^{i\theta})|^2d\theta}{4\pi\log\frac{1}{1-r}}\frac{t^2}{2}+o(t^2),\tag{12}$$

$$\left(\textit{By (10), }\limsup_{r\longrightarrow 1}\frac{\int_0^{2\pi}|b(re^{i\theta})|^2d\theta}{\log\frac{1}{1-r}}\ \textit{exists}\right).$$

Put $\Omega_t=\phi_t(\mathbb{D})$.
Now we will give the proof of Theorem 2 by using probability methods. The proof of this theorem has two steps. In the first one, we will point out the relation between the Minkowski dimension $d(t)$ of Γ_t and the spectrum of integral means $\beta(d(t),\phi_t')$ (for definition see below).

2.4 The first step of the proof.

First, we know that the image Ω_t of the unit disk $\mathbb{D}$ by the conformal map ϕ_t is a quasidisk, for t small. We recall the crucial proposition about Minkowski dimension of quasicircles (boundaries of quasidisks) .
We define

$$\beta(p,\phi')=\limsup_{r\longrightarrow 1}\frac{\log\left(\int_0^{2\pi}|\phi'(re^{i\theta})|^pd\theta\right)}{|\log(1-r)|}=\limsup_{r\longrightarrow 1}\frac{\log\left(\int_0^{2\pi}\exp\{p\mathrm{Re}b(re^{i\theta})\}d\theta\right)}{|\log(1-r)|},\ (p\in\mathbb{R})$$

be the spectrum of integral means of $\phi'=\exp b(z)$ $(z\in\mathbb{D};b\in\mathcal{B})$. In the case of the family of conformal maps $\phi_t(z)=\int_0^z e^{tb(u)}du\quad(z\in\mathbb{D};t\in\mathbb{R})$,

$$\beta(p,\phi_t')=\limsup_{r\longrightarrow 1}\frac{\log\left(\int_0^{2\pi}(\exp\{t\mathrm{Re}b(re^{i\theta})\})^pd\theta\right)}{|\log(1-r)|}=\limsup_{r\longrightarrow 1}\frac{\log\left(\int_0^{2\pi}\exp\{tp\mathrm{Re}b(re^{i\theta})\}d\theta\right)}{|\log(1-r)|}.$$

This implies that $\beta(p,\phi_t')=\beta(tp,\phi')$.

Proposition 2 *If f maps $\mathbb{D}$ conformally onto a quasidisk Ω then*

$$M.dim\partial\Omega=p$$

where p is the unique solution of $\beta(p,f')=p-1$.

Proof: See [Pom]. As a consequence of Proposition 2, we deduce the next proposition.

Proposition 3 *Let b be a Bloch function. If the spectrum of integral means of $\phi'(z) = \exp b(z)$ ($z \in \mathbb{D}$) has the development at $p = 0$:*

$$\beta(p, \phi') = ap^2 + o(p^2)$$

then the Minkowski dimension of Γ_t has the development at $t = 0$:

$$d(t) = 1 + at^2 + o(t^2).$$

Proof: We observe that $d(t) \longrightarrow 1$, as $t \longrightarrow 0$. Put $x(t) = d(t) - 1$. The proposition 2 implies that

$$\beta(d(t), \phi'_t) = d(t) - 1.$$

Since $\beta(d(t), \phi'_t) = \beta(td(t), \phi')$, we get

$$\beta(t(1 + x(t)), \phi') = x(t). \tag{13}$$

And by the assumption we have $\beta(t(1 + x(t)), \phi') = at^2(1 + x(t))^2 + o(t^2(1 + x(t))^2)$. Since $x(t) \longrightarrow 0$ as $t \longrightarrow 0$, then $t^2(1 + x(t))^2 = t^2 + o(t^2)$. This implies that

$$\beta(t(1 + x(t)), \phi') = at^2 + o(t^2) \tag{14}$$

From (13) and (14), we obtain that $x(t) = at^2 + o(t^2)$. The result follows.
Next we proceed with the second step of this proof.

2.5 The second step of the proof.

According to Proposition 3, in order to finish the proof of Theorem 2, we need to show that the family of conformal maps $\phi_t = \int_0^z e^{tb(u)} du$ where the Bloch function $b(z)$ satisfies the condition $(*)$ has the spectrum of integral means of $\phi'(z) = \exp b(z)$ expressed as $\beta(p, \phi') = ap^2 + o(p^2)$. This will be shown in the following theorem.

Theorem 3 *If b belongs to $\mathcal{B}$ and satisfies the condition $(*)$ then the spectrum of the integral means of function $\phi'(z) = \exp b(z)$ has the following development at $p = 0$:*

$$\beta(p, \phi') = \frac{1}{4} \limsup_{r \longrightarrow 1} \frac{\int_0^{2\pi} |b(re^{i\theta})|^2 d\theta}{2\pi \log(\frac{1}{1-r})} p^2 + \mathcal{O}(p^4).$$

Proof of Theorem 3: Let us give some remarks and the strategy for the proof of this theorem. First, we note that if $\gamma = \mathrm{Re}(b(0)) \neq 0$, then put $b_1(z) = b(z) - b(0)$ and we have

$$\begin{aligned}\beta(p, \phi') &= \limsup_{r \longrightarrow 1} \frac{\log \int_0^{2\pi} e^{p\gamma + p\mathrm{Re} b_1(re^{i\theta})} d\theta}{\log \frac{1}{1-r}} = \limsup_{r \longrightarrow 1} \left\{ \frac{\log \int_0^{2\pi} e^{p\mathrm{Re} b_1(re^{i\theta})} d\theta}{\log \frac{1}{1-r}} + \frac{p\gamma}{\log \frac{1}{1-r}} \right\} \\ &= \limsup_{r \longrightarrow 1} \frac{\log \int_0^{2\pi} e^{p\mathrm{Re} b_1(re^{i\theta})} d\theta}{\log \frac{1}{1-r}}.\end{aligned}$$

This says that we do not lose generality if we assume that $b(0) = 0$. Moreover, we observe that for each $r \in (0, 1)$, there exists n such that $1/2^{n+1} \leq 1 - r \leq 1/2^n$ and from (11) $\left(|b_n(e^{i\theta}) - \mathrm{Re}(b(re^{i\theta}))| \leq C\|b\|_{\mathcal{B}}, \ (r = 1 - 2^{-n}) \right)$, we deduce that

$$\beta(p, \phi') = \limsup_{r \longrightarrow 1} \frac{\log(\int_0^{2\pi} e^{pb(re^{i\theta})} d\theta)}{\log(\frac{1}{1-r})} = \limsup_{n \longrightarrow \infty} \frac{\log(\int_0^{2\pi} e^{pb_n(e^{i\theta})} d\theta)}{n \log 2}.$$

Then, Theorem 3 will follow from the estimation of the integral $\displaystyle\int_0^{2\pi} e^{pb_n(e^{i\theta})}d\theta$. The principle idea of this estimation is to make use of the exponential transformation of dyadic martingale b_n (the dyadic martingale of Reb) which is defined as a sequence

$$\begin{cases} Z_0 = \exp pb_0; \\ Z_n = \frac{\exp pb_n}{\prod_{k=0}^{n-1}\cosh(p\Delta b_k)}, n \geq 1. \end{cases}$$

Checking the condition $\mathbf{E}(Z_n|\mathcal{F}_{n-1}) = Z_{n-1}$, we see that $Z = (Z_n, \mathcal{F}_n)$ is a positive martingale. The integral $\displaystyle\int_0^{2\pi} e^{pb_n(e^{i\theta})}d\theta$ will be derived from the following equality which follows from the martingale's property that

$$\forall n \in \mathbb{N}, \quad \mathbf{E}(Z_n) = \frac{1}{2\pi}\int_0^{2\pi} \frac{\exp pb_n(e^{i\theta})}{\prod_{k=0}^{n-1}\cosh(p\Delta b_k(e^{i\theta}))}d\theta = \mathbf{E}(Z_0) = 1.$$

In other words,

$$\frac{1}{2\pi}\int_0^{2\pi} e^{pb_n(e^{i\theta}) - \log(\prod_{k=0}^{n-1}\cosh(p\Delta b_n(e^{i\theta})))}d\theta = 1. \tag{15}$$

The rest part of the estimation of $\displaystyle\int_0^{2\pi} e^{pb_n(e^{i\theta})}d\theta$ is quite simply. We just apply the following inequalities and the condition $(*)$ to (15).

$$\left| \log(\prod_{k=0}^{n-1}\cosh(p\Delta b_n)) - \frac{p^2}{2}\langle S\rangle_n^2 \right| \leq \frac{p^4}{12}\sum_{k=0}^{n-1}(\Delta b_k)^4 \leq C'p^4\|b\|_{\mathcal{B}}^2\langle S\rangle_n^2, \tag{16}$$

where C' is an absolute constant. The first inequality of (16) follows from the estimation that

$$\left| \log(\cosh(x)) - \frac{x^2}{2} \right| \leq \frac{x^4}{12}, \quad (x \in \mathbb{R}).$$

Indeed, put $g(x) = \log(\cosh(x)) - \frac{x^2}{2} - \frac{x^4}{12}$. We see that $g''(x) = -(\tanh x)^2 - x^2 \leq 0, \quad \forall x \in \mathbb{R}$. Hence, $g'(x) = \displaystyle\int_0^x g''(u)du \leq 0, \ \forall x \in \mathbb{R}$. Therefore, $g(x) = \displaystyle\int_0^x g'(u)du \leq 0, \ \forall x > 0$ and since $g(x)$ is a even function, then $g(x) \leq 0, \ \forall x \in \mathbb{R}$. Similarly, put $h(x) = \log(\cosh(x)) - \frac{x^2}{2} + \frac{x^4}{12}$. We observe that $h''(x) = -(\tanh x)^2 + x^2 \geq 0, \quad \forall x \in \mathbb{R}$ because $|\tanh x| \leq |x|, \quad \forall x \in \mathbb{R}$. Analogously, we obtain that $\forall x \in \mathbb{R}, \quad h(x) \geq 0$.
Besides, (11) ($\forall \xi \in \mathbb{T}, \ |\Delta b_n(\xi)| \leq C\|b\|_{\mathcal{B}}$) implies that

$$\sum_{k=0}^{n-1}(\Delta b_k)^4 = \sum_{k=0}^{n-1}(\Delta b_k)^2(\Delta b_k)^2 \leq C^2\|b\|_{\mathcal{B}}^2\sum_{k=0}^{n-1}(\Delta b_k)^2 = C^2\|b\|_{\mathcal{B}}^2\langle S\rangle_n^2.$$

Then the second one of (16) follows.
Finally, we'll apply the following lemma to conclude that for p small

$$\beta(p,\phi') = \limsup_{n\longrightarrow\infty} \frac{\log\left(\int_0^{2\pi} e^{pb_n(e^{i\theta})}d\theta\right)}{n\log 2} = \frac{p^2}{4}\limsup_{r\longrightarrow 1}\frac{\int_0^2 |b(re^{i\theta})|^2(\theta)d\theta}{2\pi\log\frac{1}{1-r}} + \mathcal{O}(p^4)$$

Lemma 1 *Let b be a Bloch function and $\langle S\rangle_n^2$ be the square function of the dyadic martingale b_n of Re(b). Then*

$$\limsup_{n\longrightarrow\infty}\frac{\int_0^{2\pi}\langle S\rangle_n^2(e^{i\theta})d\theta}{n\log 2}=\limsup_{r\longrightarrow 1}\frac{\int_0^{2\pi}|b(re^{i\theta})|^2d\theta}{2\log\frac{1}{1-r}}\leq\pi\|b\|_{\mathcal{B}}^2.$$

Proof: Recall $\tilde{b}=\mathrm{Re}b$ and b_n is a dyadic martingale of $\tilde{b}$. We have:

$$\|b_n\|_2^2=\int_0^{2\pi}b_n^2(\theta)d\theta=\int_0^{2\pi}\sum_{k=0}^{n-1}(\Delta b_k(\theta))^2d\theta=\int_0^{2\pi}\langle S\rangle_n^2(\theta)d\theta.$$

The second equality follows from Proposition 5.4.5 [Gra] and the third one follows from the definition of the square function of the dyadic martingale b_n. Moreover, the fact that $|b_n(\theta)-\tilde{b}(re^{i\theta})|\leq C\|b\|_{\mathcal{B}}$ if $r=1-2^{-n}$ (see (11)) implies that:

$$\Bigg|\,\|b_n\|_2-\|\tilde{b}(re^{i\theta})\|_2\,\Bigg|\leq\|b_n(e^{i\theta})-\tilde{b}(re^{i\theta})\|_2\leq 2\pi(C\|b\|_{\mathcal{B}}).$$

Therefore if we divide both sides by $(n\log 2)^{1/2}$ of the above inequalities and take the limit as $n\longrightarrow\infty$, then we obtain:

$$\lim_{j\longrightarrow\infty}\left(\frac{\int_0^{2\pi}(b_n)^2d\theta}{n\log 2}\right)^{1/2}-\left(\frac{\int_0^{2\pi}(\tilde{b}((1-2^{-n})e^{i\theta}))^2d\theta}{n\log 2}\right)^{1/2}=0. \tag{17}$$

By Proposition 1, $\dfrac{\int_0^{2\pi}(\tilde{b}((1-2^{-n})e^{i\theta}))^2d\theta}{n\log 2}$ is bounded and then by (17) $\dfrac{\int_0^{2\pi}(b_n)^2d\theta}{n\log 2}$ is also bounded. Moreover since the function x^2 is continuous uniformly on some compact set of $[0,+\infty)$, then (17) implies that

$$\lim_{j\longrightarrow\infty}\frac{\int_0^{2\pi}(b_n)^2d\theta}{n\log 2}-\frac{\int_0^{2\pi}(\tilde{b}((1-2^{-n})e^{i\theta}))^2d\theta}{n\log 2}=0.$$

Thus,

$$\limsup_{j\longrightarrow\infty}\frac{\int_0^{2\pi}(b_n)^2d\theta}{j\log 2}=\limsup_{j\longrightarrow\infty}\frac{\int_0^{2\pi}(\tilde{b}((1-2^{-n})e^{i\theta}))^2d\theta}{n\log 2}.$$

Then,

$$\limsup_{n\longrightarrow\infty}\frac{\int_0^{2\pi}\langle S\rangle_n^2(\theta)d\theta}{n\log 2}=\limsup_{r\longrightarrow 1}\frac{\int_0^{2\pi}(\tilde{b}(re^{i\theta}))^2d\theta}{\log(\frac{1}{1-r})}\quad (r=1-2^{-n}).$$

Furthermore, since b is holomorphic in the unit disk $\mathbb{D}$ and by Proposition 1, we have:

$$\limsup_{r\longrightarrow 1}\frac{\int_0^{2\pi}(\mathrm{Re}b(re^{i\theta}))^2d\theta}{\log(\frac{1}{1-r})}=\limsup_{r\longrightarrow 1}\frac{\int_0^{2\pi}|b(re^{i\theta})|^2d\theta}{2\log(\frac{1}{1-r})}\leq\pi\|b\|_{\mathcal{B}}^2.$$

The lemma is proven.
The proof of Theorem 3 remains the main step: that is to estimate the integral $\int_0^{2\pi}e^{pb_n(e^{i\theta})}d\theta$.
The main step of the proof.

Put $\epsilon_n(\theta) = \begin{cases} \frac{\log(\prod_{k=0}^{n-1}\cosh(p\Delta b_n(e^{i\theta})))-\frac{p^2}{2}\langle S\rangle_n^2(e^{i\theta})}{\frac{p^2}{2}\langle S\rangle_n^2(e^{i\theta})}, & \text{if } \langle S\rangle_n^2(e^{i\theta}) \neq 0 \\ 0, & \text{otherwise} \end{cases}$, $(\theta \in [0, 2\pi])$. This says that

$$\log\left(\prod_{k=0}^{n-1}\cosh(p\Delta b_n(e^{i\theta}))\right) = \frac{p^2}{2}\langle S\rangle_n^2(e^{i\theta})\left(1+\epsilon_n(\theta)\right), \tag{18}$$

where $|\epsilon_n(\theta)| \leq C'p^2\|b\|_{\mathcal{B}}^2$ by (16).

Put $I_n = \frac{1}{2\pi}\int_0^{2\pi}\langle S\rangle_n^2(\theta)d\theta$. By (18), (15) is equivalent to

$$\frac{1}{2\pi}\int_0^{2\pi}\exp\left\{pb_n(e^{i\theta}) - \frac{p^2}{2}\langle S\rangle_n^2(e^{i\theta})(1+\epsilon_n(\theta))\right\}d\theta = 1.$$

By subtraction and adding the term $\frac{p^2}{2}I_n(1+\epsilon_n(\theta))$, we can rewrite the preceding equality as follows

$$\frac{1}{2\pi}\int_0^{2\pi}\exp\left\{pb_n(e^{i\theta}) - \frac{p^2}{2}I_n(1+\epsilon_n(\theta)) - \frac{p^2}{2}(\langle S\rangle_n^2(e^{i\theta}) - I_n)(1+\epsilon_n(\theta))\right\}d\theta = 1.$$

Remark that I_n is a number, so we can take the term $\exp\{\frac{p^2}{2}I_n\}$ out of the above integral, then the equality turns out to be

$$\frac{1}{2\pi}\int_0^{2\pi}\exp\left\{pb_n(e^{i\theta}) - \frac{\epsilon_n(\theta)p^2}{2}I_n - \frac{p^2}{2}(\langle S\rangle_n^2(e^{i\theta}) - I_n)(1+\epsilon_n(\theta))\right\}d\theta = \exp\left(\frac{p^2}{2}I_n\right).$$

Put $I = \frac{1}{2\pi}\int_0^{2\pi}\exp\left\{pb_n(e^{i\theta}) - \frac{\epsilon_n(\theta)p^2}{2}I_n - \frac{p^2}{2}(\langle S\rangle_n^2(e^{i\theta}) - I_n)(1+\epsilon_n(\theta))\right\}d\theta$. Next, we will estimate the integral I.

Combining the condition $(*)$ $\left|\langle S\rangle_n^2(\xi) - I_n\right| \leq n\delta(n)$ with the fact that $|\epsilon_n| \leq C'p^2\|b\|_{\mathcal{B}}^2$, we have: $\left|(1+\epsilon_n(\theta))(\langle S\rangle_n^2(\xi) - I_n)\right| \leq (1+C'p^2\|b\|_{\mathcal{B}}^2)n\delta(n)$. Then, this implies that:

$$\exp\left\{-C'p^4\|b\|_{\mathcal{B}}^2 I_n - \frac{n\delta(n)}{2}p^2(1+C'p^2\|b\|_{\mathcal{B}}^2)\right\}\frac{1}{2\pi}\int_0^{2\pi}e^{pb_n(e^{i\theta})}d\theta \leq I$$

and

$$I \leq \exp\left\{C'p^4\|b\|_{\mathcal{B}}^2 I_n + \frac{n\delta(n)}{2}p^2(1+C'p^2\|b\|_{\mathcal{B}}^2)\right\}\frac{1}{2\pi}\int_0^{2\pi}e^{pb_n(e^{i\theta})}d\theta.$$

Replacing I by $\exp\left(\frac{p^2}{2}I_n\right)$ and then taking logarithm of two sides of the above inequalities, we deduce that

$$\log\left(\int_0^{2\pi}e^{pb_n(e^{i\theta})}d\theta\right) - \frac{n\delta(n)}{2}p^2(1+C'p^2\|b\|_{\mathcal{B}}) - C'p^4\|b\|_{\mathcal{B}}^2 I_n - \log(2\pi) \leq \frac{p^2}{2}I_n$$

and

$$\frac{p^2}{2}I_n \leq \log\left(\int_0^{2\pi}e^{pb_n(e^{i\theta})}d\theta\right) + \frac{n\delta(n)}{2}p^2(1+C'p^2\|b\|_{\mathcal{B}}) + C'p^4\|b\|_{\mathcal{B}}^2 I_n - \log(2\pi).$$

Next, if we divide two both sides of the inequalities by $n\log 2$, we then obtain the inequalities

$$\frac{p^2}{2}\frac{I_n}{n\log 2} \geq \frac{\log\left(\int_0^{2\pi} e^{pb_n(e^{i\theta})}d\theta\right)}{n\log 2} - \left(p^2(1+C'p^2\|b\|^2_{\mathcal{B}})\frac{n\delta(n)}{2n\log 2} + C'p^4\|b\|^2_{\mathcal{B}}\frac{I_n}{n\log 2} + \frac{\log(2\pi)}{n\log 2}\right)$$

and

$$\frac{p^2}{2}\frac{I_n}{n\log 2} \leq \frac{\log\left(\int_0^{2\pi} e^{pb_n(e^{i\theta})}d\theta\right)}{n\log 2} + \left(p^2(1+C'p^2\|b\|^2_{\mathcal{B}})\frac{n\delta(n)}{2n\log 2} + C'p^4\|b\|^2_{\mathcal{B}}\frac{I_n}{n\log 2} - \frac{\log(2\pi)}{n\log 2}\right).$$

Taking the lim sup as n tends to ∞ of these inequalities, then we get

$$\left(\frac{p^2}{2} - C'p^4\|b\|^2_{\mathcal{B}}\right)\limsup_{n\longrightarrow\infty}\frac{I_n}{n\log 2} \leq \limsup_{n\longrightarrow\infty}\frac{\log\left(\int_0^{2\pi} e^{pb_n(e^{i\theta})}d\theta\right)}{n\log 2} \leq \left(\frac{p^2}{2} + C'p^4\|b\|^2_{\mathcal{B}}\right)\limsup_{n\longrightarrow\infty}\frac{I_n}{n\log 2}.$$

Finally, we obtain the estimation

$$\left|\limsup_{n\longrightarrow\infty}\frac{\log\left(\int_0^{2\pi} e^{pb_n(e^{i\theta})}d\theta\right)}{n\log 2} - \frac{p^2}{2}\limsup_{n\longrightarrow\infty}\frac{I_n}{n\log 2}\right| \leq C'p^4\|b\|^2_{\mathcal{B}}\limsup_{n\longrightarrow\infty}\frac{I_n}{n\log 2}, \tag{19}$$

where $\limsup_{n\longrightarrow\infty}\frac{I_n}{n\log 2} = \limsup_{n\longrightarrow\infty}\frac{\int_0^{2\pi}\langle S\rangle_n^2(\theta)d\theta}{2\pi n\log 2} = \limsup_{r\longrightarrow 1}\frac{\int_0^{2\pi}|b(re^{i\theta})|^2 d\theta}{4\pi\log(\frac{1}{1-r})} \leq \frac{\|b\|^2_{\mathcal{B}}}{2} < +\infty$ by Lemma 1. Thus, the estimation (19) gives us the desired formula for the spectrum of integral means

$$\beta(p,\phi') = \limsup_{r\longrightarrow 1}\frac{\log\left(\int_0^{2\pi} e^{pb(e^{i\theta})}d\theta\right)}{\log(\frac{1}{1-r})} = \frac{p^2}{4}\limsup_{r\longrightarrow 1}\frac{\int_0^{2\pi}|b(re^{i\theta})|^2 d\theta}{2\pi\log(\frac{1}{1-r})} + \mathcal{O}(p^4),$$

as p tends to zero. This finishes the proof of Theorem 3.
From Theorem 3 and Proposition 3, we conclude Theorem 2. For the sake of completeness of this part, we will give a non-trivial example for Bloch function which satisfies condition $(*)$

2.6 An example with constant square function.

First, we define the independent Bernoullian random variables ε_n on $\partial\mathbb{D}$ by the formula

$$\varepsilon_n(e^{2\pi ix}) = \begin{cases} -1, & x_n = 0 \text{ or } 3, \\ 1, & x_n = 1 \text{ or } 2, \end{cases} \qquad (n = 1, 2, ...)$$

where x_n denotes the 4-adic nth digit of $x \in [0,1]$.

Proposition 4 *For any bounded sequence of a real numbers $\{a_n\}$, the 4-adic martingale $S_n = \sum_{k=1}^{n} a_k\varepsilon_k$ is a dyadic martingale (if considered as dyadic).*

Proof: See [Mak].

Let $\{a_k\}$ be a bounded sequence of real number, then $\limsup_{n\longrightarrow\infty}\frac{\sum_{k=1}^n a_k^2}{n} = \alpha < +\infty$. By Proposition 4, there exists a Bloch function b which generates the dyadic martingale S_n.

Let $\phi_t(z) = \displaystyle\int_0^z e^{tb(u)}du$: these are conformal mappings from $\mathbb{D}$ onto Ω_t and the Minkowski dimension of $\Gamma_t = \partial\Omega_t$ has the following development at 0:

$$\begin{aligned} \text{M.dim}(\Gamma_t) &= 1 + \limsup_{r\longrightarrow 1} \frac{\int_0^{2\pi} |b(re^{i\theta})|^2 d\theta}{4\pi \log \frac{1}{1-r}} \frac{t^2}{2} + o(t^2) \\ &= 1 + \frac{\alpha}{2\log 2} t^2 + o(t^2). \end{aligned} \tag{20}$$

Indeed, since $\Delta S_k = a_k \varepsilon_k$ then $\langle S \rangle_n^2 = \displaystyle\sum_{k=1}^{n} a_k^2$ is a constant square function. Thus, certainly the square function $\langle S \rangle_n^2$ satisfies the condition $(*)$. Besides, we have

$$\limsup_{r\longrightarrow 1} \frac{\int_0^{2\pi} |b(re^{i\theta})|^2 d\theta}{2\pi \log(\frac{1}{1-r})} = 2 \limsup_{n\longrightarrow\infty} \frac{\int_0^{2\pi} \langle S \rangle_n^2(\theta) d\theta}{2\pi n \log 2} = 2 \limsup_{n\longrightarrow\infty} \frac{\langle S \rangle_n^2}{n \log 2} = \frac{2\alpha}{\log 2}, \quad (r = 1 - 2^n).$$

Then, (20) follows from Theorem 2.

3 Counter-example.

In this part we show that Mc Mullen's property does not hold for all $b \in \mathcal{B}$. The counter-example we construct is reminiscent of Kahane's construction of a non-Smirnov domain.

3.1 Kahane measure and its Herglotz transform.

3.1.1 Kahane measure

First of all, let us recall the construction of Kahane measure. Denote by ω_0 the interval $[0, 1]$ and by ω_j one of intervals of form 4-adic $[p4^{-j}, (p + 1)4^{-j}]$ contained in ω_0. We construct simultaneously a sequence of measure μ_j and their supports E_j as follow:

μ_0 is the Lebesgue measure on interval ω_0;

μ_j is proportional to the Lebesgue measure on each ω_j.

We denote by $D_j(\omega_j)$ its density on a given interval ω_j and its support E_j is the union of intervals ω_j where $D_j(\omega_j) \neq 0$. In order to obtain μ_{j+1} from μ_j, we divide each interval $\omega = \omega_j$ of rank j contained in E_j into four equal subintervals $\omega^1, \omega^2, \omega^3, \omega^4$ of rank $j + 1$ and put

$$\begin{aligned} D_{j+1}(\omega^1) = D_{j+1}(\omega^4) = D_j(\omega) - 1, \\ D_{j+1}(\omega^2) = D_{j+1}(\omega^3) = D_j(\omega) + 1. \end{aligned}$$

Put $\mu = \displaystyle\lim_{j\longrightarrow\infty} \mu_j$ and $E = \displaystyle\bigcap_{j=0}^{\infty} E_j$. We call this measure μ Kahane's measure.

There is another way to define the set E. Recall the independent Bernoullian random variables ε_k on $\partial\mathbb{D}$ (defined in 2): put $\Sigma_j(e^{2\pi ix}) = \displaystyle\sum_{k=1}^{j} \varepsilon_k(e^{2\pi ix})$ and let N be the first number such that $1 + \displaystyle\sum_{k=1}^{j} \varepsilon_k(e^{2\pi ix}) = 0$ $(x \in [0, 1])$. By the definition of D_k, we have :

$$\forall x \in [0, 1], \ \ D_0(x) = 1; \ \ D_k(x) = (D_{k-1}(x) + \varepsilon_k(e^{2\pi ix}))1_{E_{k-1}}(x).$$

Therefore,

$$D_k(x) = \left(\left(\left(\left(1+\varepsilon_1(e^{2\pi ix}) \right) 1_{E_0}(x)+\varepsilon_2(e^{2\pi ix}) \right) 1_{E_1}(x)+...+ \right) 1_{E_{k-2}}(x)+\varepsilon_k(e^{2\pi ix}) \right) 1_{E_{k-1}}(x), (x \in [0,1]).$$

Since $E_0 \supset E_1 \supset ... \supset E_{k-1}$ then $1_{E_0}...1_{E_{k-1}} = 1_{E_{k-1}}$, therefore

$$D_k(x) = (1 + \Sigma_k(e^{2\pi ix}))1_{E_{k-1}}(x).$$

This implies that the support of D_k: $E_k = E_{k-1} \cap \{1 + \Sigma_k > 0\}$. Then,

$$E_k = \{1 + \Sigma_1 > 0, ..., 1 + \Sigma_k > 0\}, \ (k = 1, 2, ...).$$

Moreover, for $x \in [0, 1]$

$$\begin{aligned} D_k(x) &= (1 + \Sigma_k(e^{2\pi ix}))1_{E_{k-1}}(x) \\ &= (1 + \Sigma_k(e^{2\pi ix}))1_{E_k}(x) + (1 + \Sigma_k(e^{2\pi ix}))1_{E_{k-1}\setminus E_k}(x) \\ &= (1 + \Sigma_k(e^{2\pi ix}))1_{E_k}(x). \end{aligned}$$

Because on the set $E_{k-1} \setminus E_k$ we have $1 + \Sigma_k(x) = 0$.

In his paper [Kah], Kahane showed that the set $E = \bigcap_{k=0}^{\infty} E_k$ (support of the measure μ) has a null Lebesgue measure. Therefore this measure is totally singular.

3.1.2 Herglotz transform of Kahane measure

Let $b(z)$ be Herglotz transform of Kahane measure μ: that is

$$b(z) = \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\mu(\theta).$$

Kahane has proved that $b \in \mathcal{B}$. Put $\Lambda_j(e^{2\pi ix}) = 1 + \Sigma_j(e^{2\pi ix})$ and

$$S_j(e^{2\pi ix}) = \Lambda_{j\wedge N}(e^{2\pi ix}) = \begin{cases} 1 + \Sigma_j(e^{2\pi ix}), & \text{if } x \in \{N > j\} = E_j \\ 0, & \text{otherwise} \end{cases} , \ (x \in [0, 1]).$$

Similarly to the example of the square constant function in 2.6 above, Λ_j is a dyadic martingale (if consider as dyadic). By the construction of μ, $\{N = j\} = \bigcup \omega_j = E_{j-1} \setminus E_j \in \mathcal{F}_j$, where ω_j is an interval 4-adic of rank $j - 1$ i.e dyadic of rank j. Therefore N is a stopping time with respect to the σ-algebra $\{\mathcal{F}_j, j \geq 0\}$ (defined above). Thus, $S_j = \Lambda_{N\wedge j}$ is a dyadic martingale as well. Moreover, we have the following lemma.

Lemma 2 *S_j is the dyadic martingale of the Bloch function Re(b).*

Proof: Indeed, we recall $h(\theta)$ the cumulative distribution function of the Kahane measure μ, (i.e. $h(\varphi) = \mu(\{\frac{\varphi}{2\pi} > 0\})$ $(\varphi \in [0, 2\pi])$ and $h(0) = 0$. We observe that for $z \in \mathbb{D}$

$$b(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\varphi} + z}{e^{i\varphi} - z} h'(\varphi) d\varphi = \frac{1}{2\pi} \int_0^{2\pi} \left(1 + 2 \sum_{n=1}^{\infty} e^{-in\varphi} z^n \right) h'(\varphi) d\varphi. \tag{21}$$

By the Schwartz integral formula and Im$b(0) = 0$, we have

$$b(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\varphi} + z}{e^{i\varphi} - z} \mathrm{Re}b(e^{i\varphi}) d\varphi = \frac{1}{2\pi} \int_0^{2\pi} \left(1 + 2 \sum_{n=1}^{\infty} e^{-in\varphi} z^n \right) \mathrm{Re}b(e^{i\varphi}) d\varphi. \tag{22}$$

From (21),(22) we obtain $\displaystyle\int_0^{2\pi} e^{-in\varphi}(\mathrm{Re}b(e^{i\varphi}) - h'(\varphi))d\varphi = 0 \ \ (n = 0, 1, 2, ...)$. Since the sequence $\{e^{in\theta}\}(n = 0, 1, 2, ...)$ is a basic in $L^2([0, 2\pi])$, then

$$\mathrm{Re}b(e^{i\varphi}) - h'(\varphi) = 0 \qquad \text{in } L^2([0, 2\pi]).$$

Thus, $\mathrm{Re}b(e^{i\varphi}) = h'(\varphi)$ a.e in $[0, 2\pi]$. We observe that for each subarc 4-adic $\omega_j = [\frac{\varphi_0}{2\pi}, \frac{\varphi_0}{2\pi} + \frac{\varphi}{2\pi}]$ of rank j of the interval $[0, 1]$,

$$\begin{aligned}
\frac{\mu_j(\omega_j)}{|\omega_j|} &= \frac{1}{|\omega_j|}\int_{\omega_j} D_j(x)dx \\
&= \frac{1}{|\omega_j|}\int_{\omega_j} (1 + \Sigma_j(e^{2\pi ix}))1_{E_j}dx \\
&= \Lambda_{j\wedge N}(e^{2\pi ix})1_{\omega_j}(x) = S_j(e^{2\pi ix})|_{\omega_j},
\end{aligned}$$

while $\dfrac{\mu_j(\omega_j)}{|\omega_j|} = \dfrac{h(\varphi + \varphi_0) - h(\varphi_0)}{|\varphi|}$. Therefore,

$$S_j(e^{2\pi ix})|_{\omega_j} = \frac{h(\varphi + \varphi_0) - h(\varphi_0)}{|\varphi|} = \frac{1}{|\omega_j|}\int_{\omega_j} \mathrm{Re}b(e^{2\pi ix})dx = (\mathrm{Re}b)_{\omega_j}.$$

It means that S_j is the dyadic martingale S_j of the Bloch function $\mathrm{Re}b$. Now, let us state concretely the second result of this paper.

3.2 Statement of Theorem 4.

Let μ be Kahane's measure and $b(z)$ its Herglotz transform. We recall that Γ_t is the image of the unit circle $\mathbb{T}$ by the conformal map $\phi_t(z)$ which is defined as $\phi_t'(z) = e^{tb(z)}$, t small enough. If a family of conformal maps $\phi_t(z) = \int_0^z e^{tb(u)}du, \ (z \in \mathbb{D}; b \in \mathcal{B})$ satisfies (8) with Hausdorff dimension replaced by Minkowski dimension, then

$$\mathrm{M.dim}(\Gamma_t) = 1 + \limsup_{r \longrightarrow 1} \frac{\int_0^{2\pi} |b(re^{i\theta})|^2 d\theta}{4\pi \log \frac{1}{1-r}} \frac{t^2}{2} + o(t^2). \tag{23}$$

Theorem 4 *The behaviour of the curve Γ_t differs with the sign of t:*
In the case of negative t, the singular property of the Kahane's measure μ (the density function of the probability measure μ is non negative and zero almost everywhere) makes $\phi_t' \in H^1$. This is equivalent to the rectifiability of Γ_t and then $H.dim(\Gamma_t) = M.dim(\Gamma_t) \equiv 1$.
On the other hand, in the case of positive t, Γ_t is a fractal curve and its Minkowski dimension satisfies the following inequality:

$$d(t) \geq 1 + \frac{t^2}{8 \log 2}, \quad \forall t > 0 \ \textit{ small enough},$$

as a consequence the family of conformal map $(\phi_t), t > 0$ gives a counter-example to (23).

Next, we'll give the proof of this theorem.

3.3 Proof of Theorem 4

First of all, we will use the singularity of Kahane measure to show that in the case of small negative t, $\mathrm{H.dim}(\Gamma_t) = \mathrm{M.dim}(\Gamma_t) \equiv 1$.

3.3.1 Negative t.

We recall now the two theorems on $H^p(p>0)$ functions and then we'll show how they imply the first part of Theorem 4. Let us introduce some notions. Given a function $f(z) \not\equiv 0$ of class $H^p(p>0)$. Let (a_n) (may be finite, or even empty) be the sequence zeroes of the function f. A function of the form

$$B(z) = z^m \prod_n \frac{|a_n|}{a_n} \frac{a_n - z}{1 - \overline{a_n} z}$$

is called a Blaschke product. A singular inner function is a function of the form

$$S(z) = \exp\left\{ -\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} d\mu(t) \right\},$$

where $\mu(t)$ be a bounded non-decreasing singular function ($\mu'(t) = 0$ a.e). And an outer function of class H^p is a function of form

$$F(z) = e^{i\gamma} \exp\left\{ \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} \log|f(e^{i\theta})| d\theta \right\},$$

where γ is a real number, $|f(e^{i\theta})| \in L^p([0,2\pi])$.

Theorem 5 *(Canonical factorization theorem). Every function $f(z) \not\equiv 0$ of class $H^p(p>0)$ has a unique factorization of the form $f(z) = B(z)S(z)F(z)$ where $B(z)$ is a Blaschke product, $S(z)$ is a singular inner function and $F(z)$ is an outer function of class H^p. Conversely, every such product $B(z)S(z)F(z)$ belongs to H^p.*

Proof: See [Dur].

Theorem 6 *Let $f(z)$ maps the unit disk $\mathbb{D}$ conformally onto a Jordan domain Ω. Then the boundary $\partial\Omega$ is rectifiable if and only if $f' \in H^1$.*

Proof: See [Dur].
Since t small enough and $b(z)$ is a Bloch function, then by Becker univalence criterion the maps $\phi_t(z)$ maps conformally the unit disk $\mathbb{D}$ onto a quasidisk Ω_t. And its derivative has the form

$$\phi_t'(z) = \exp\left\{ t \int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} d\mu(\theta) \right\},$$

where $t<0$ and μ is a positive singular measure i.e the density function $h'(\theta)$ of Kahane measure μ is non negative and zero almost everywhere on $[0,2\pi]$ (mentioned above). Then, Theorem 5 yields $\phi_t' \in H^1$.
Since $\phi_t' \in H^1$ is equivalent to the rectifiability of the boundary Γ_t by Theorem 6, then obviously

$$\text{H.dim}(\Gamma_t) = \text{M.dim}(\Gamma_t) \equiv 1. \tag{24}$$

The first part of Theorem 4 follows.
Now, we'll go to the main part of the proof of Theorem 4: the case of small positive t.

3.3.2 Positive t.

We want to show that $d(t) \geq 1 + \frac{t^2}{8\log 2}$, $t>0$ small. Analogously to section 2, in order to prove this, we need to show that the spectrum of integral means $\beta(p,\phi')$ where $\phi' = \exp b(z)$ satisfies the following inequality

$$\beta(p,\phi') \geq \frac{p^2}{8\log 2}, \quad p>0 \text{ small}. \tag{25}$$

In addition, from the fact that $|S_j(e^{i\theta}) - \mathrm{Re}(b(re^{i\theta}))| \leq C\|b\|_{\mathcal{B}}$, $(r = 1-2^{-j})$, see (11), we have

$$\beta(p,\phi') = \limsup_{r\longrightarrow 1} \frac{\int_0^{2\pi} e^{p\mathrm{Re}b(re^{i\theta})}d\theta}{\log\frac{1}{1-r}} = \limsup_{j\longrightarrow\infty} \frac{\int_0^{2\pi} e^{pS_j(e^{i\theta})}d\theta}{j\log 2}.$$

This leads us to estimate the integral $\displaystyle\int_{\mathbb{T}} e^{pS_j(e^{i\theta})}d\theta$, $(S_j = \Lambda_{j\wedge N})$, $p > 0$ small. The difficult point is that S_j is not a sum of independent random variables. However we can go around this difficulty by using the stopping time of the random walk argument of the dyadic martingale S_j which will be introduced in the following.

3.3.3 Random walk argument.

Let us describe this random walk on graph. On the lattice $\mathbb{Z}^+ \times \mathbb{Z}$, we consider that a particle moves in the direction parallel to two diagonals of the unit square. We denote the individual steps generically by $\varepsilon_1, \varepsilon_2, ..., \varepsilon_n$ with the probability $p = \frac{1}{2}$ (defined above in 2.6) and the position of the particle by $\Sigma_1, \Sigma_2, ..., \Sigma_n$. According to the assumption of this dyadic martingale, the particle will stop as it reaches to the horizontal axis $y = 0$ on the lattice.
We denote the event $\{$ at epoch n the particle is at the position r $\}$ by $\{\Sigma_n = r\}$ and we can write the event $\{N > k\}$ by $\{1 + \Sigma_1 > 0, ..., 1 + \Sigma_k > 0\}$ and then by $\{\Sigma_1 \geq 0, ..., \Sigma_k \geq 0\}$. We need the following lemma to obtain the inequality (25).

Lemma 3 *For a random walk $\Sigma_n = \varepsilon_1 + \varepsilon_2 + ... + \varepsilon_n$, where ε_k are Bernoulli independent random variables with the probability $p = \frac{1}{2}$, we have:*

$$P(\Sigma_1 \geq 0, \Sigma_2 \geq 0, ..., \Sigma_{2n} \geq 0) = P(\Sigma_{2n} = 0) = \frac{C_{2n}^n}{2^{2n}}.$$

Moreover by Stirling's formula $P(N > 2n) \simeq \dfrac{1}{\sqrt{2n}}$.

Proof: See [Fel].
Furthermore, we remark that for each positive integer k, $\{N > 2k+1\} = \{N > 2k\}$. Indeed, $\{\Sigma_1 \geq 0, ..., \Sigma_{2k} \geq 0\} = \{\Sigma_1 \geq 0, ..., \Sigma_{2k} \geq 0, \Sigma_{2k+1} \geq 0\} \cap \{\Sigma_1 \geq 0, ..., \Sigma_{2k} \geq 0, \Sigma_{2k+1} < 0\}$. By the assumption of the stopping time, the particle will stop as it reaches to the axis $y = 0$, hence $\{\Sigma_1 \geq 0, ..., \Sigma_{2k} \geq 0, \Sigma_{2k+1} < 0\} = \emptyset$. Thus, $\{\Sigma_1 \geq 0, ..., \Sigma_{2k} \geq 0\} = \{\Sigma_1 \geq 0, ..., \Sigma_{2k} \geq 0, \Sigma_{2k+1} \geq 0\}$.
Now we proceed to the main step of the proof of Theorem 4.

3.3.4 The main step of the proof.

We'll estimate the integral $\displaystyle\int_{\mathbb{T}} e^{pS_j(e^{i\theta})}d\theta$. First we note that on the set $\{N \leq j\}$ $S_j(e^{i\theta}) = \Lambda_{j\wedge N}(e^{i\theta}) = 0$, then

$$\begin{aligned} \frac{1}{2\pi}\int_{\mathbb{T}} e^{pS_j(e^{i\theta})}d\theta &= \frac{1}{2\pi}\int_{\{N>j\}} e^{pS_j(e^{i\theta})}d\theta + \frac{1}{2\pi}\int_{\{N\leq j\}} e^{pS_j(e^{i\theta})}d\theta \\ &= \frac{1}{2\pi}\int_{\{N>j\}} e^{pS_j(e^{i\theta})}d\theta + P(\{N \leq j\}), \end{aligned} \tag{26}$$

where $\displaystyle\frac{1}{2\pi}\int_{\{N>j\}} e^{pS_j(e^{i\theta})}d\theta = \frac{1}{2\pi}\int_{\{N>j\}} e^{p(1+\Sigma_j(e^{i\theta}))}d\theta$. We observe that

$$\frac{1}{2\pi}\int_{\{N>j\}} e^{p(1+\Sigma_j(e^{i\theta}))}d\theta = \frac{1}{2\pi}\int_{\mathbb{T}} e^{p(1+\Sigma_j(e^{i\theta}))}d\theta - \frac{1}{2\pi}\int_{\{N\leq j\}} e^{p(1+\Sigma_j(e^{i\theta}))}d\theta.$$

Since $\Sigma_j = \sum_{k=1}^{j} \varepsilon_k$ where ε_k with $k = 1, 2, ...$ are the independent random variables, then the integral

$$\frac{1}{2\pi}\int_{\mathbb{T}} e^{p(1+\Sigma_j(e^{i\theta}))} d\theta = e^p \prod_{k=1}^{j} \mathbf{E}(e^{p\varepsilon_k}) = e^p \prod_{k=1}^{j} \cosh p = e^p (\cosh p)^j.$$

Besides, the integral $\int_{\{N\leq j\}} e^{p(1+\Sigma_j(e^{i\theta}))} d\theta$ can be rewritten as:

$$\int_{\{N\leq j\}} e^{p(1+\Sigma_j(e^{i\theta}))} d\theta = \sum_{k=1}^{j} \int_{\{N=k\}} e^{p(1+\Sigma_j(e^{i\theta}))} d\theta.$$

The fact that $1 + \Sigma_k(e^{i\theta})$ is equal to zero on each set $\{N = k\}$ makes the value of the integral $\sum_{k=1}^{j} \int_{\{N=k\}} e^{p(1+\Sigma_j(e^{i\theta}))} d\theta$ unchanged if we divide the integrand $e^{p(1+\Sigma_j(e^{i\theta}))}$ by the term $e^{1+\Sigma_k(e^{i\theta})}$. Thus we have:

$$\begin{aligned}\sum_{k=1}^{j} \int_{\{N=k\}} e^{p(1+\Sigma_j(e^{i\theta}))} d\theta &= \sum_{k=1}^{j} \int_{\{N=k\}} e^{p(1+\Sigma_j(e^{i\theta})-1-\Sigma_k(e^{i\theta}))} d\theta \\ &= \sum_{k=1}^{j} \int_{\{N=k\}} e^{p(\Sigma_j(e^{i\theta})-\Sigma_k(e^{i\theta}))} d\theta.\end{aligned}$$

In addition, if we rewrite the integral $\int_{\{N=k\}} e^{p(\Sigma_j(e^{i\theta})-\Sigma_k(e^{i\theta}))} d\theta$ as $\int_{\mathbb{T}} 1_{\{N=k\}} e^{p(\Sigma_j(e^{i\theta})-\Sigma_k(e^{i\theta}))} d\theta$, then by the independence of two random variables $1_{\{N=k\}}$ and $e^{p(\Sigma_j-\Sigma_k)}$ it follows that

$$\begin{aligned}\frac{1}{2\pi}\int_{\mathbb{T}} 1_{\{N=k\}} e^{p(\Sigma_j(e^{i\theta})-\Sigma_k(e^{i\theta}))} d\theta &= P(\{N = k\}) \mathbf{E}(e^{p(\Sigma_j(e^{i\theta})-\Sigma_k(e^{i\theta}))}) \\ &= P(\{N = k\})(\cosh p)^{j-k}.\end{aligned}$$

Hence we obtain

$$\begin{aligned}\frac{1}{2\pi}\int_{\{N>j\}} e^{pS_j(e^{i\theta})} d\theta &= e^p \cosh(p)^j \left(1 - \sum_{k=1}^{j} \frac{P(\{N = k\})}{(\cosh p)^k e^p}\right) \\ &\geq e^p (\cosh p)^j \left(1 - \sum_{k=1}^{j} P(\{N = k\})\right), \quad (p > 0) \\ &= e^p (\cosh p)^j P(\{N > j\}), \quad (p > 0). \end{aligned} \tag{27}$$

The inequality above follows from the fact that for $p > 0$ $(\cosh p)^k e^p \geq 1$, $k = 1, 2, .., j$. From (26), (27) and Jensen's inequality, we deduce

$$\begin{aligned}\log\left(\int_{\mathbb{T}} e^{pS_j(e^{i\theta})} d\theta\right) &\geq \frac{1}{2}\log\left(2\int_{\{N>j\}} e^{pS_j(e^{i\theta})} d\theta\right) + \frac{1}{2}\log\left(4\pi P(\{N \leq j\})\right) \\ &\geq \frac{p}{2} + \log(4\pi) + \frac{1}{2}\log(\cosh(p)^j) + \frac{1}{2}\log(P(\{N > j\})) + \frac{1}{2}\log(P(\{N \leq j\}))\end{aligned}$$

By Lemma 3: $\log(P(N > j)) \simeq -\dfrac{\log j}{2}$ and $\log(P(N \leq j)) \simeq -\dfrac{1}{\sqrt{j}}$ as $j \longrightarrow \infty$, thus when we divide the above inequality by $j \log 2$ and take the lim sup as $j \longrightarrow \infty$, we deduce that

$$\beta(p, \phi') \geq \limsup_{j\longrightarrow\infty} \frac{\log(\cosh(p)^j)}{2j \log 2} = \frac{\log\cosh(p)}{2\log 2}, \quad p > 0.$$

Moreover, the inequality $\log\cosh(x) \geq \frac{x^2}{2} - \frac{x^4}{12}$, $(x > 0)$ (proved in 2) implies that $\log\cosh(x) \geq \frac{x^2}{4}$ for $x > 0$ small enough, which implies (25): $\beta(p, \phi') \geq \frac{p^2}{8\log 2}$, $p > 0$ small. As a consequence of (25), the spectrum of integral means $\beta(d(t), \phi'_t)$ of the family of the conformal maps $\phi'_t(z) = \exp tb(z)$ satisfies the following inequality:

$$\beta(d(t), \phi'_t) = \beta(td(t), \phi') \geq \frac{t^2 d(t)^2}{8\log 2}, \quad t > 0 \text{ small},$$

where $d(t) = \text{M.dim}(\Gamma_t) \geq 1$.
Finally, by Proposition 2: $d(t) = \beta(d(t), \phi'_t) + 1$, we deduce that :

$$d(t) \geq 1 + \frac{t^2}{8\log 2}, \quad t > 0 \text{ small}. \tag{28}$$

This means that (23) fails for the family of conformal map (ϕ_t), $t > 0$ because if this family holds for (23) then the fact that

$$\limsup_{r\longrightarrow 1} \frac{\int_0^{2\pi} |b(re^{i\theta})|^2 d\theta}{\log\frac{1}{1-r}} = 2\limsup_{r\longrightarrow 1} \frac{\int_0^{2\pi} |\text{Re}b(re^{i\theta})|^2 d\theta}{\log\frac{1}{1-r}} = 0 \tag{29}$$

which follows from the following results would contradict (28). Theorem 4 is proven.

Theorem 7 *Let $\langle S\rangle_j^2$ be the square function of the dyadic martingale S_j of Reb (b defined above) and a real positive p, then there exist positive constants $M_1, M_2, K_1, K_2, T_1, T_2$ do not depend on j such that:*
If $p > 1$

$$M_1 j^{(p-1)/2} \leq \frac{1}{2\pi}\int_0^{2\pi} (\langle S\rangle_j^2)^{p/2} d\theta \leq M_2 j^{(p-1)/2};$$

If $p = 1$

$$K_1 \log j \leq \frac{1}{2\pi}\int_0^{2\pi} (\langle S\rangle_j^2)^{p/2} d\theta \leq K_2 \log j;$$

If $p < 1$

$$T_1 \leq \frac{1}{2\pi}\int_0^{2\pi} (\langle S\rangle_j^2)^{p/2} d\theta \leq T_2.$$

Proof: First we'll show that

$$\frac{1}{2\pi}\int_0^{2\pi} (\langle S\rangle_j^2(\theta))^{p/2} d\theta = \sum_{k=1}^{j-1}((k+1)^{p/2} - k^{p/2})P(\{N > k\}) \tag{30}$$

and then we'll prove that there exist positive constant A_1, A_2 do not depend on j such that

$$A_1 \sum_{n=1}^{l} \frac{1}{(2n)^{(3-p)/2}} \leq \int_0^{2\pi} (\langle S\rangle_j^2)^{p/2} d\theta \leq A_2 \sum_{n=1}^{l} \frac{1}{(2n)^{(3-p)/2}}. \tag{31}$$

The proof will follow from the estimation of the sum $\sum_{n=1}^{l} \frac{1}{(2n)^{(3-p)/2}}$. We first separate the unit circle into two sets $\{N > j\}$ and $\{N \leq j\}$, then

$$\frac{1}{2\pi}\int_0^{2\pi} (\langle S\rangle_j^2)^{p/2} d\theta = \frac{1}{2\pi}\int_{\{N>j\}} (\langle S\rangle_j^2)^{p/2} d\theta + \frac{1}{2\pi}\int_{\{N\leq j\}} (\langle S\rangle_j^2)^{p/2} d\theta.$$

We observe that on the set $\{N > j\}$, $\langle S\rangle_j^2 = j$, hence $\frac{1}{2\pi}\int_{\{N>j\}} (\langle S\rangle_j^2)^{p/2} d\theta = j^{p/2} P(\{N > j\})$.
Besides,

$$\int_{\{N\leq j\}} (\langle S\rangle_j^2)^{p/2} d\theta = \sum_{k=1}^{j} \int_{\{N=k\}} (\langle S\rangle_j^2)^{p/2} d\theta.$$

Note that $\langle S\rangle_j^2 = k$ on $\{N = k\}$. This implies that

$$\frac{1}{2\pi}\int_{\{N\leq j\}} (\langle S\rangle_j^2)^{p/2} d\theta = \sum_{k=1}^{j} \frac{1}{2\pi}\int_{\{N=k\}} (\langle S\rangle_j^2)^{p/2} d\theta = \sum_{k=1}^{j} k^{p/2} P(\{N = k\}).$$

By using summation by parts, we have

$$\sum_{k=1}^{j} k^{p/2} P(\{N = k\}) = \sum_{k=1}^{j-1} ((k+1)^{p/2} - k^{p/2}) P(\{N > k\}) - j^{p/2} P(\{N > j\}).$$

This implies (30). We observe that if $p \geq 2$ then

$$\frac{1}{2\pi}\int_0^{2\pi} (\langle S\rangle_j^2(\theta)) d\theta \geq \sum_{k=1}^{j-1} (p/2) k^{(p-2)/2} P(\{N > k\})$$

and

$$\frac{1}{2\pi}\int_0^{2\pi} (\langle S\rangle_j^2(\theta)) d\theta \leq \sum_{k=1}^{j-1} (p/2)(k+1)^{(p-2)/2} P(\{N > k\}),$$

and since $(k+1)^{(p-2)/2} \leq e^{(p-2)/2} k^{(p-2)/2}, k = 1, 2, \ldots$ then

$$\frac{1}{2\pi}\int_0^{2\pi} (\langle S\rangle_j^2(\theta)) d\theta \leq e^{(p-2)/2} \sum_{k=1}^{j-1} (p/2) k^{(p-2)/2} P(\{N > k\}).$$

Thus,

$$\frac{p}{2}\sum_{k=1}^{j-1} k^{(p-2)/2} P(\{N > k\}) \leq \frac{1}{2\pi}\int_0^{2\pi} (\langle S\rangle_j^2)^{p/2} d\theta \leq \frac{pe^{(p-2)/2}}{2} \sum_{k=1}^{j-1} k^{(p-2)/2} P(\{N > k\}).$$

If $p < 2$ we have the inverse inequality

$$\frac{p}{2}\sum_{k=1}^{j-1} k^{(p-2)/2} P(\{N > k\}) \geq \frac{1}{2\pi}\int_0^{2\pi} (\langle S\rangle_j^2)^{p/2} d\theta \geq \frac{pe^{(p-2)/2}}{2} \sum_{k=1}^{j-1} k^{(p-2)/2} P(\{N > k\}),$$

Using the remark in 3.3.3 that $P(\{N > 2n+1\}) = P(\{N > 2n\})$, therefore without lost generality we assume that $j = 2(l+1)$.
If $p \geq 2$ then

$$\begin{aligned} p/2 + p\sum_{n=1}^{l} (2n)^{(p-2)/2} P(\{N > 2n\}) &\leq \frac{1}{2\pi}\int_0^{2\pi} (\langle S\rangle_j^2)^{p/2} d\theta \\ &\leq \frac{pe^{(p-2)/2}}{2} \left((1 + e^{(p-2)/2}) \sum_{n=1}^{l} (2n)^{(p-2)/2} P(\{N > 2n\}) + 1/2 \right), \end{aligned}$$

If $p < 2$ then

$$p/2 + p\sum_{n=1}^{l}(2n)^{(p-2)/2}P(\{N > 2n\}) \geq \frac{1}{2\pi}\int_0^{2\pi}(\langle S\rangle_j^2)^{p/2}d\theta$$
$$\geq \frac{pe^{(p-2)/2}}{2}\left((1+e^{(p-2)/2})\sum_{n=1}^{l}(2n)^{(p-2)/2}P(\{N > 2n\}) + 1/2 \right),$$

By Lemma 3, there exist absolute positive constants C_1, C_2 such that

$$C_1\frac{1}{\sqrt{2n}} \leq P(N > 2n) = P(\{\Sigma_{2n} = 0\}) \leq C_2\frac{1}{\sqrt{2n}}.$$

This implies that

$$C_1\sum_{n=1}^{l}\frac{1}{(2n)^{(3-p)/2}} \leq \sum_{n=0}^{l}(2n)^{(p-2)/2}P(\{N > 2n\}) \leq C_2\sum_{n=1}^{l}\frac{1}{(2n)^{(3-p)/2}}.$$

This implies (31). Now we observe that if $p \geq 3$ the function $f(x) = \dfrac{1}{(2x)^{(3-p)/2}}$ is increasing on $[1,\infty)$, then

$$2^{(p-3)/2} + \int_1^l \frac{1}{(2x)^{(3-p)/2}}dx \leq \sum_{n=1}^{l+1}\frac{1}{(2n)^{(3-p)/2}} \leq \int_1^{l+1}\frac{1}{(2x)^{(3-p)/2}}dx,$$

where $\int_1^l \frac{1}{(2x)^{(3-p)/2}}dx = \frac{1}{p-1}\left((j-2)^{(p-1)/2} - 2^{(p-1)/2} \right)$ and $\int_1^{l+1}\frac{1}{(2x)^{(3-p)/2}}dx = \frac{1}{p-1}\Big((j)^{(p-1)/2} - 2^{(p-1)/2} \Big)$.

If $p < 3$ the function $f(x) = \dfrac{1}{2n^{(3-p)/2}}$ is decreasing on $[1,\infty)$, then :
if $\dfrac{3-p}{2} < 1 \Longleftrightarrow p > 1$ then

$$\int_1^{l+1}\frac{1}{(2x)^{(3-p)/2}}dx \leq \sum_{n=1}^{l}\frac{1}{(2n)^{(3-p)/2}} \leq \int_1^l \frac{1}{(2x)^{(3-p)/2}}dx + \frac{1}{2^{(3-p)/2}},$$

where $\int_1^l \frac{1}{(2x)^{(3-p)/2}}dx = \frac{1}{p-1}\left((j-2)^{(p-1)/2} - 2^{(p-1)/2} \right)$ and $\int_1^{l+1}\frac{1}{(2x)^{(3-p)/2}}dx = \frac{1}{p-1}\Big((j)^{(p-1)/2} - 2^{(p-1)/2} \Big)$;
if $\dfrac{3-p}{2} = 1 \Longleftrightarrow p = 1$ then

$$\frac{1}{2}\int_1^{l+1}\frac{1}{x}dx \leq \sum_{n=1}^{l}\frac{1}{2n} \leq \frac{1}{2}\int_1^l \frac{1}{x}dx + \frac{1}{2},$$

where $\int_1^{l+1} \frac{1}{x} dx = \log(j) - \log 2$ and $\int_1^{l} \frac{1}{x} dx = \log(j-2) - \log 2$;
if $\frac{3-p}{2} > 1 \Longleftrightarrow p < 1$ then $\sum_{n=1}^{l} \frac{1}{(2n)^{(3-p)/2}}$ converges as $j \longrightarrow \infty$ because

$$\int_1^{l+1} \frac{1}{(2x)^{(3-p)/2}} dx \leq \sum_{n=1}^{l} \frac{1}{(2n)^{(3-p)/2}} \leq \int_1^{l} \frac{1}{(2x)^{(3-p)/2}} dx + \frac{1}{2^{(3-p)/2}},$$

where $\int_1^{l} \frac{1}{(2x)^{(3-p)/2}} dx = \frac{1}{p-1} \Bigg((j-2)^{(p-1)/2} - 2^{(p-1)/2} \Bigg)$ and $\int_1^{l+1} \frac{1}{(2x)^{(3-p)/2}} dx = \frac{1}{p-1} \Bigg((j)^{(p-1)/2} - 2^{(p-1)/2} \Bigg)$ converges as $j \longrightarrow \infty$.
Now we can go to the conclusion that there exist positive constants $M_1, M_2, K_1, K_2, T_1, T_2$ do not depend on j such that:
If $p > 1$

$$M_1 j^{(p-1)/2} \leq \frac{1}{2\pi} \int_0^{2\pi} (\langle S \rangle_j^2)^{p/2} d\theta \leq M_2 j^{(p-1)/2};$$

If $p = 1$

$$K_1 \log j \leq \frac{1}{2\pi} \int_0^{2\pi} (\langle S \rangle_j^2)^{p/2} d\theta \leq K_2 \log j;$$

If $p < 1$

$$T_1 \leq \frac{1}{2\pi} \int_0^{2\pi} (\langle S \rangle_j^2)^{p/2} d\theta \leq T_2.$$

The theorem is proven.

Corollary 2 *Let b be the Bloch function (defined above) and a real positive p, then*

$$\limsup_{r \longrightarrow 1} \frac{\int_0^{2\pi} |Reb(re^{i\theta})|^p d\theta}{(\log \frac{1}{1-r})^{p/2}} = 0$$

Proof: The proof will be given as follows. First of all, we'll show that for $p > 0$

$$\limsup_{j \longrightarrow \infty} \frac{\int_0^{2\pi} |\mathrm{Re} b((1-2^{-j})e^{i\theta})|^p d\theta}{(j \log 2)^{p/2}} = \limsup_{j \longrightarrow \infty} \frac{\int_0^{2\pi} |S_j|^p d\theta}{(j \log 2)^{p/2}}. \tag{32}$$

Then we'll estimate $\frac{\int_0^{2\pi} |S_j|^p d\theta}{(j \log 2)^{p/2}}$ by using the fact that: for $1 < p < \infty$ (see [Bur]) there exist absolute positive constants b_p and B_p such that

$$b_p \|\langle S \rangle_j^2\|_{(p/2)}^2 \leq \|S_j\|_p \leq B_p \|\langle S \rangle_j^2\|_{(p/2)}^2$$

and for $0 < p \leq 1$ (see [Gan]) there also exists a positive absolute constant ν_p such that $\|S_j\|_p \leq \nu_p \|\langle S \rangle_j^2\|_{(p/2)}^2$, where the square function $\langle S \rangle_j^2 = \sum_{k=1}^{n} (\Delta S_k)^2$. Then the proof will follow by Theorem 7. That is the main idea of the proof.
First, let us prove (32). The fact that $|S_j(\theta) - \mathrm{Re} b(re^{i\theta})| \leq C\|b\|_{\mathcal{B}}$ if $r = 1 - 2^{-j}$ (see (11)) implies that for $p \geq 1$

$$\Bigg| \|S_j\|_p - \|\mathrm{Re} b(re^{i\theta})\|_p \Bigg| \leq \|S_j - \mathrm{Re} b(re^{i\theta})\|_p \leq 2\pi(C\|b\|_{\mathcal{B}}).$$

Therefore if we divide both sides by $(j\log 2)^{1/2}$ of the above inequalities and take the limit as j tends to ∞, then we obtain

$$\lim_{j\longrightarrow\infty}\left(\frac{\int_0^{2\pi}|S_j|^p d\theta}{(j\log 2)^{p/2}}\right)^{1/p}-\left(\frac{\int_0^{2\pi}|\mathrm{Re}b((1-2^{-j})e^{i\theta})|^p d\theta}{(j\log 2)^{p/2}}\right)^{1/p}=0. \tag{33}$$

According to Corollary 1 $\dfrac{\int_0^{2\pi}|\mathrm{Re}b((1-2^{-j})e^{i\theta})|^p d\theta}{(j\log 2)^{p/2}}$ is bounded and then by (33) $\dfrac{\int_0^{2\pi}|S_j|^p d\theta}{(j\log 2)^{p/2}}$ is also bounded. Moreover since the function x^p is continuous uniformly on some compact set of $[0,+\infty)$, then (33) implies that

$$\lim_{j\longrightarrow\infty}\frac{\int_0^{2\pi}|S_j|^p d\theta}{(j\log 2)^{p/2}}-\frac{\int_0^{2\pi}|\mathrm{Re}b((1-2^{-j})e^{i\theta})|^p d\theta}{(j\log 2)^{p/2}}=0$$

Thus,

$$\limsup_{j\longrightarrow\infty}\frac{\int_0^{2\pi}|S_j|^p d\theta}{(j\log 2)^{p/2}}=\limsup_{j\longrightarrow\infty}\frac{\int_0^{2\pi}|\mathrm{Re}b((1-2^{-j})e^{i\theta})|^p d\theta}{(j\log 2)^{p/2}}.$$

In the case of $0<p\leq 1$, again the fact that $|S_j(\theta)-\mathrm{Re}b(re^{i\theta})|\leq C\|b\|_{\mathcal{B}}$ if $r=1-2^{-j}$ (see (11)) implies that

$$\left|\int_0^{2\pi}|S_j|^p d\theta-\int_0^{2\pi}|\mathrm{Re}b((1-2^{-j})e^{i\theta})|^p d\theta\right|\leq\int_0^{2\pi}|S_j-\mathrm{Re}b((1-2^{-j})e^{i\theta})|^p d\theta\leq 2\pi(C\|b\|_{\mathcal{B}})^p.$$

Analogously, if we devide the above inequalities by $(j\log 2)^{p/2}$ and take the limit as j tends to ∞, then we have

$$\lim_{j\longrightarrow\infty}\frac{\int_0^{2\pi}|S_j|^p d\theta}{(j\log 2)^{p/2}}-\frac{\int_0^{2\pi}|\mathrm{Re}b((1-2^{-j})e^{i\theta})|^p d\theta}{(j\log 2)^{p/2}}=0$$

which implies that

$$\limsup_{j\longrightarrow\infty}\frac{\int_0^{2\pi}|S_j|^p d\theta}{(j\log 2)^{p/2}}=\limsup_{j\longrightarrow\infty}\frac{\int_0^{2\pi}|\mathrm{Re}b((1-2^{-j})e^{i\theta})|^p d\theta}{(j\log 2)^{p/2}}.$$

Then (32) follows.

According to Theorem 7, if we divide the integral $\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi}(\langle S\rangle_j^2(\theta))^{p/2}d\theta$ by $(j\log 2)^{p/2}$ and let $j\longrightarrow\infty$, then we have

$$\limsup_{j\longrightarrow\infty}\frac{\int_0^{2\pi}(\langle S\rangle_j^2(\theta))^{p/2}d\theta}{(j\log 2)^{p/2}}=0.$$

This finishes the proof.

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